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Stability Considerations of Discrete Event Dynamic Systems Based on STP and Boolean Networks Concept

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Abstract: Nowadays there are many event-driven types of control systems, e.g., manufacturing systems, industrial and welfare robots, networked control systems (popularly known as IoT), and so forth. In this paper, the security and stability problems of discrete-event dynamic systems are studied based on semi-tensor product (STP) and Boolean networks concept. The permutation and logical matrix expressions of such structural nonlinear systems are introduced. The relation to Perron-Frobenius theory and irreducible M-matrix is also described. Simple numerical examples are shown to illustrate recurrent dynamic systems.

Keywords: Discrete event systems; permutation matrices; semi-tensor product; logical matrices; M-matrices

1. INTRODUCTION

At present, there are many event-driven types of control systems [1-3], e.g., manufacturing systems, industrial and welfare robots, networked control systems [4, 5], and so on. In this paper, the security and stability problems of discrete-event dynamic systems (DEDSs) are studied based on semi-tensor product (STP) and Boolean networks concept [6, 7]. In the analytical process, permutation and logical matrix expressions of such structural nonlinear systems will be introduced. The relation to Perron-Frobenius theory and irreducible M-matrix will be described. Although the theory of strucural mapping and graph expressions of DEDSs have been known, e.g. [8, 9], we think that the quantitative evaluation for the securities and stability of the systems may be difficult. Thus we will use permutation matrix and STP expressions in the systems analysis.

2. MATRIX EXPRESSION OF DEDSs

Finite-state and discrete-event dynamic systems can be written as the following multistage processes [1]:

$$\begin{aligned} &\boldsymbol{x}(t_{h+1}) = \boldsymbol{f}(\boldsymbol{x}(t_h), \boldsymbol{e}(t_h)) \qquad (1)\\ &h \in \mathbb{N} := \{0, 1, 2, \ldots 1, N\}, \end{aligned}$$

where $\boldsymbol{x}(\cdot)$, $\boldsymbol{e}(\cdot)$, and $\boldsymbol{f}(\cdot,\cdot)$ are state variables, event (control) signals, and a transition function, respectively. In this paper, each variable is considered as

$$\boldsymbol{x}(t_h) \in \boldsymbol{\mathcal{X}} \subset \mathbb{Z}^n,\ \boldsymbol{e}(t_h) \in \boldsymbol{\mathcal{E}} \subset \mathbb{Z}^m,\ \boldsymbol{f} : \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{X}},$$

where $\mathbb{Z}$ is a finite set of integers[2],

Of course, the states and events may be non-numerical (qualitative) situations in practice. However, they can be considered as ordered sets, and thus the ordered sets will be replaced by integer numbers. For example, when considering a DEDS as shown in Fig. 1 [3] [10], the following states and events can be defined:

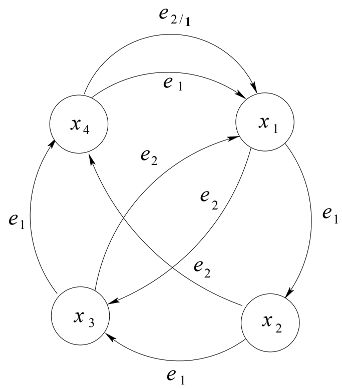

Fig. 1 State transition graph for a DEDS with four places and state-values.

$$\boldsymbol{\mathcal{X}} = \{x_1, x_2, x_3, x_4\}, \quad \boldsymbol{\mathcal{E}} = \{e_1, e_2, e_\phi\},$$

where e_ϕ means no action.

When the clossing terms of (1) are not considered here[4], DEDSs will be written by using a permutation matrix [11-14] as

$$x(t_{h+1}) = \boldsymbol{P}(\boldsymbol{e}(t_k)) \cdot x(t_h), \qquad (2)$$

where $\boldsymbol{P}(\boldsymbol{e}(t_h)) \in \boldsymbol{\mathcal{P}}$. Here, $\boldsymbol{\mathcal{P}} \subset \mathbb{Z}^{n\times n}$ is a set of permutation matrices in which only one nonzero element $p_{ij} > 0$ is permitted in the row and column elements of $\boldsymbol{P} = (p_{ij})$. As is obvious, the number of the positive element p_{ij} in the matrix $\boldsymbol{P}$ is n. When considering $p_{ij} = 1$, those matrices are said to be (0,1)-permutation matrix.

Let $\sigma = k_1 k_2 \cdots k_n$ be a permutation of $\{1, 2, \cdots, n\}$ [14]. $\boldsymbol{P} = [p_{ij}]$ can be defined in regard to the j-th row

† Yoshifumi Okuyama is the presenter of this paper.

[1] Although the equation is a clock-driven (time-driven) expression, event-driven processes can also be expressed in this form.

[2] Hereafter, it will be considered as a set of natural numbers. When considering only $\mathbb{Z} \Rightarrow \mathcal{D} := \{0, 1\}$, the proesss becomes a Boolean (i.e., automata) network.

[3] Although this example is only a vending machine, it will correspond to some industrial machine in practice.

[4] Not only algebraic operations but also logical operations, e.g., $\wedge$, $\vee$, and $1 + 1 = 1$ are not considered here.

as follows:

$$p_{ij} = \begin{cases} 1, & \text{if } j = k_i, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In regard to a DEDS as shown in Fig. 1, a (circulant) permutation matrix $\boldsymbol{P}$ is, for example, given below:

$$\boldsymbol{P}(\boldsymbol{e}(t_h)) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4}$$

for $\sigma = k_1 k_2 k_3 k_4 = 4123$. Note that it can also be defined for the i-th column as:

$$p_{ij} = \begin{cases} 1, & \text{if } i = k_j, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Based on the above consideration, (2) can be expanded as follows:

$$\boldsymbol{x}(t_N) = \left(\prod_{h=0}^{N-1} \boldsymbol{P}(\boldsymbol{e}(t_h)) \right) \boldsymbol{x}(t_0). \tag{6}$$

Here, the following is approved:

$$\boldsymbol{P}(\boldsymbol{e}(t_N)) \in \boldsymbol{\mathcal{P}} \quad \Rightarrow \quad \prod_{h=0}^{N-1} \boldsymbol{P}(\boldsymbol{e}(t_h)) \in \boldsymbol{\mathcal{P}}, \tag{7}$$

and the converse holds.

3. SEMI-TENSOR PRODUCT

The STP approach becomes common, especially, in Chinese researchers with regard to Boolean networks (i.e., automata theory) [6, 7, 15]. In these literatures, the definition of STP of matrices is given below.

[Definition] With respect to $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{Z}^{p \times q}$

(1) The left STP

$$\boldsymbol{A} \ltimes \boldsymbol{B} = (\boldsymbol{A} \otimes \boldsymbol{I}_{\alpha/n})(\boldsymbol{B} \otimes \boldsymbol{I}_{\alpha/p}), \tag{8}$$

(2) The right STP

$$\boldsymbol{A} \rtimes \boldsymbol{B} = (\boldsymbol{I}_{\alpha/n} \otimes \boldsymbol{A})(\boldsymbol{I}_{\alpha/p} \otimes \boldsymbol{B}) \tag{9}$$

where[5]

$\alpha = \text{lcm}(n, p)$, $\boldsymbol{I}_m$: $m \times m$ identity matrix

$\otimes$: tensor product (Kronecker product).

When considering the left STP of $\boldsymbol{F} \in \mathbb{Z}^{n \times mn}$ and $\boldsymbol{e} \in \mathbb{Z}^{m \times 1}$, the following expression can be obtained:

$$\boldsymbol{F} \ltimes \boldsymbol{e} = (\boldsymbol{F} \otimes \boldsymbol{I}_{\alpha/mn})(\boldsymbol{e} \otimes \boldsymbol{I}_{\alpha/n}). \tag{10}$$

In this case, $\alpha = \text{lcm}(mn, m) = mn$. Therefore, it can be given as follows:

$$\boldsymbol{F} \ltimes \boldsymbol{e} = \boldsymbol{F}(\boldsymbol{e} \otimes \boldsymbol{I}_m). \tag{11}$$

[5] $\text{lcm}(n, p)$ means a least common multiple of n and p.

By using the above expression, the mathematical model of DEDSs (2) is written for any h-th event (control signal) as:

$$\boldsymbol{x}(t_{h+1}) = (\boldsymbol{F} \ltimes \boldsymbol{e}(t_h)) \ltimes \boldsymbol{x}(t_h) = (\boldsymbol{F} \times \boldsymbol{\delta}_m^k) \ltimes \boldsymbol{x}(t_h), \tag{12}$$

where

$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 & \boldsymbol{F}_2 & \cdots & \boldsymbol{F}_m \end{bmatrix}$, $\boldsymbol{F}_k \in \mathcal{P} \subset \mathbb{Z}^{n \times n}$,

$\boldsymbol{\delta}_m^k$: k-th column of the identity matrix $\boldsymbol{I}_m$,

$\boldsymbol{x}(t_h) \in \mathbb{Z}^n \ \ (1 \le k \le m)$.

Note that $\boldsymbol{\delta}_m^k$ correspods to a *basis vector* in the matrix theory.

Moreover, the following relation can be obtained:

$$\boldsymbol{F} \ltimes \boldsymbol{\delta}_m^k = \boldsymbol{F}(\boldsymbol{\delta}_m^k \otimes \boldsymbol{I}_m) = \boldsymbol{F}_k. \tag{13}$$

Therefore, (12) can be written as follows:

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{F}_k \ltimes \boldsymbol{x}(t_h) = \boldsymbol{F}_k \cdot \boldsymbol{x}(t_h), \tag{14}$$

where

$$\boldsymbol{F}_k \in \{\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_m\}.$$

In this paper, the state $\boldsymbol{x}(t_h)$ will not be restricted within $\mathcal{D} := \{0, 1\}$ as descrribed in Boolean networks.

On the other hand, (12) can also be written as:

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{F} \ltimes (\boldsymbol{e}(t_h) \ltimes \boldsymbol{x}(t_h)) = \boldsymbol{F}(\boldsymbol{\delta}_m^k \otimes \boldsymbol{x}(t_h), \quad (1 \le k \le m), \tag{15}$$

based on the associative law of STP. Thus, the following expression is obtained:

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{F}(\boldsymbol{\delta}_m^k \otimes \boldsymbol{x}(t_h) = \boldsymbol{F} \cdot \begin{bmatrix} \boldsymbol{x}_1(t_h) \\ \boldsymbol{x}_2(t_h) \\ \vdots \\ \boldsymbol{x}_m(t_h) \end{bmatrix}, \tag{16}$$

where $\boldsymbol{x}_j(\cdot) \in \mathbb{Z}^n$, and

$$\boldsymbol{x}_j(t_h) = \begin{cases} \boldsymbol{x}(t_h) & \text{when } j = k \\ \boldsymbol{0} & \text{when } j \ne k. \end{cases}$$

4. LOGICAL MATRICES AND THEIR OPERATIONS

The transition matrices $\boldsymbol{F}_k$ and $\boldsymbol{F}$ in (14) and (16) can be expressed by the following logical matrix[6]:

$$\boldsymbol{F}_k, \ \boldsymbol{F} \ \sim \ \boldsymbol{L}_n = [\boldsymbol{\delta}_n^{i_1} \ \boldsymbol{\delta}_n^{i_2} \ \cdots \ \boldsymbol{\delta}_n^{i_{mn}}].$$

[6] Although the logical matrix is defined with repect to a column of the identity matrix $\boldsymbol{I}_n$ as described in the literature[6, 14], we think that it can also be defined by a row of $\boldsymbol{I}_n$ as given below:

$$\boldsymbol{L}_n = \begin{bmatrix} \boldsymbol{\sigma}_n^{j_1} \\ \boldsymbol{\sigma}_n^{j_2} \\ \vdots \\ \boldsymbol{\sigma}_n^{j_{mn}} \end{bmatrix}, \quad \boldsymbol{L}_n = \boldsymbol{\sigma}_n \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_{mn} \end{bmatrix}, \quad (j_l \le n),$$

where

$\boldsymbol{\sigma}_n^l$: l-th row of the identity matrix $\boldsymbol{I}_n$.

The above corresponds to $\sigma = 4123$ in (4). This matrix means a transition 'from' a place. On the other hand, (17) will be a transition 'to' a place.

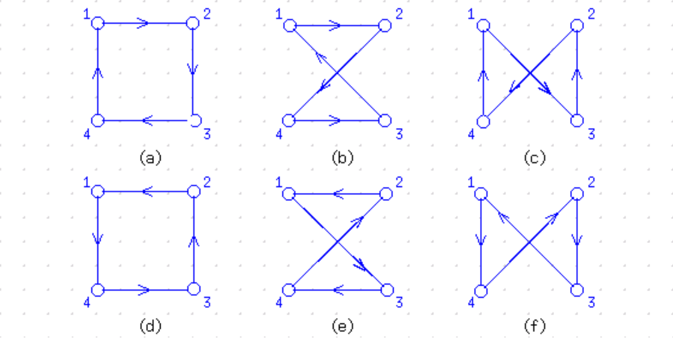

Fig. 2 Cycles of length $n = 4$.

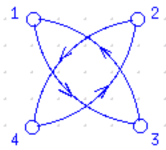

Fig. 3 A reducible graph for $n = 4$.

It can also be written as

$$\boldsymbol{L}_n = \boldsymbol{\delta}_n[i_1\ i_2\ \cdots\ i_{mn}], \quad (i_l \leq n). \tag{17}$$

In regard to a circulant system shown in (4), the following expression can be obtained:

$$\boldsymbol{L}_4 = \boldsymbol{\delta}_4 \begin{bmatrix} 2 & 3 & 4 & 1 \end{bmatrix}. \tag{18}$$

As for fourth-order periodic systems as shown in Fig. 2 ($(n-1)!$ directed graphs[26]), the following logical matrices are obtained[7] :

$$\begin{cases} \text{(a)}: & \boldsymbol{\delta}_4[2\ 3\ 4\ 1] \\ \text{(b)}: & \boldsymbol{\delta}_4[2\ 4\ 1\ 3] \\ \text{(c)}: & \boldsymbol{\delta}_4[3\ 4\ 2\ 1] \\ \text{(d)}: & \boldsymbol{\delta}_4[4\ 1\ 2\ 3] \\ \text{(e)}: & \boldsymbol{\delta}_4[3\ 1\ 4\ 2] \\ \text{(f)}: & \boldsymbol{\delta}_4[4\ 3\ 1\ 2]. \end{cases} \tag{19}$$

Of course, based on $\boldsymbol{F}$ in (16), the following 4×8 ($m = 2$) matrix can be defined, e.g.,

$$\boldsymbol{F} = \boldsymbol{L}_4 = \boldsymbol{\delta}_4[\ 2\ 3\ 4\ 1\ 3\ 4\ 2\ 1\ ] \tag{20}$$

for (a) and (c). However, in the following, we will consider $m = 1$ cases (i.e., $n \times n$ permutation matrices $\boldsymbol{P}$).

In regard to these matrices the following (multiplicative) operations can be defined:

$$\begin{aligned} \boldsymbol{L}_{n1} \cdot \boldsymbol{L}_{n2} &= \boldsymbol{\delta}_n[i_1\ i_2 \cdots\ i_n] \cdot \boldsymbol{\delta}_n[j_1\ j_2\ \cdots\ j_n] \\ &= \boldsymbol{\delta}_n[i_{j_1}\ i_{j_2}\ \cdots\ i_{j_n}] \end{aligned} \tag{21}$$

[7]It should be noted that the order of numbers in the logical matrices is different from the order of node numbers in Fig. 2.

An example of the multiplication of logical matrices can be given as [8]:,

$$\begin{aligned} \boldsymbol{L}_{31} &= \boldsymbol{\delta}_4 \begin{bmatrix} 3 & 4 & 1 & 2 \end{bmatrix} \cdot \boldsymbol{\delta}_4 \begin{bmatrix} 2 & 3 & 4 & 1 \end{bmatrix} \\ &= \boldsymbol{\delta}_4 \begin{bmatrix} 4 & 1 & 2 & 3 \end{bmatrix}. \end{aligned} \tag{22}$$

Here, the original expression is

$$\boldsymbol{L}_{31} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

When we write $\boldsymbol{P} = \boldsymbol{\delta}_4[2\ 3\ 4\ 1]]$ and $\boldsymbol{P}_c =_4 [3\ 4\ 1\ 2]$, the following relations can be obtained:

$$\boldsymbol{P}_c \cdot \boldsymbol{P} = \boldsymbol{P}^T = \boldsymbol{P}^{-1}, \tag{23}$$

and

$$\det \boldsymbol{P} = \det \boldsymbol{P}^T = -1, \quad \det \boldsymbol{P}_c = 1. \tag{24}$$

Here, $\boldsymbol{P}_c$ correspods to a reducible graph as shown in Fig. 3. Note that in regard to the general permutation matrix $\boldsymbol{P}$ the following is valid:

$$\det \boldsymbol{P} = \pm 1. \tag{25}$$

Next, let us consider the multiplication by a vector $\boldsymbol{x} = [x_1\ x_2\ x_3\ x_4]^T$. The logical matrix expression can be given as follows:

$$\boldsymbol{\delta}_4 \begin{bmatrix} 2 & 3 & 4 & 1 \end{bmatrix} \cdot [\hat{x}_1\ \hat{x}_2\ \hat{x}_3\ \hat{x}_4]^T = [\hat{x}_4\ \hat{x}_1\ \hat{x}_2\ \hat{x}_3]^T \tag{26}$$

In general,

$$\begin{aligned} &\boldsymbol{\delta}_n[i_1\ i_2 \cdots\ i_n] \cdot [\hat{x}_1\ \hat{x}_2\ \cdots\ \hat{x}_n]^T \\ &= [\hat{x}_{i_1}\ \hat{x}_{i_2}\ \cdots\ \hat{x}_{i_n}]^T. \end{aligned} \tag{27}$$

Here, $\hat{x}_j$ $(j = 1, 2, \cdots, n)$ denotes a value (simply 'token') of each place in $[x_1\ x_2\ \cdots\ x_n]^T$, which corresponds to the initial state of x_j. In the Boolean networks, these values are written as $\hat{x}_j \in \mathcal{D}^n$.

In the analysis of these logical DEDSs, it is important that the node y as shown in Fig. 4 (a) does not accept two (or more) signals at a time. The addition rule will be prohibited. On the other hand, signals from the node x as shown in (b) may be allowed.

[8]These operations may be hard to understand in the mathematical expression. Here, the multiplication is given using calored characters. From 2 of the right matrix to the 2nd element of left matrix 4, the 1st element 4 of the result matrix is obtained. Next, from 3 to the 3rd element 1, the 2nd element 1 is obtained, and so on.

On the computer programming, the above is easily realized by using C-language, e.g.,

```
for(i=1;i<=4;i++){
     for(j=1;j<=4;j++){
          if(i==d1[j]) dd[j]=d2[i];
     }
}
```

Here, `d1[*]` and `d2[*]` are the numbers in the right and left logical matrices (22), respectively, and `dd[*]` is the number in the resulted matrix.

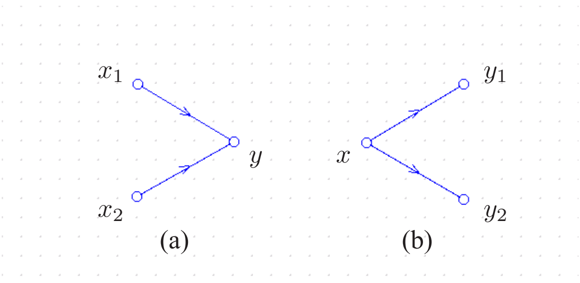

Fig. 4 Addition prohibited and drawable connections.

5. RECURRENT DYNAMIC SYSTEMS

When using the expression of (27), dynamic systems (2) can be written as

$$[x_1(t_{h+1})\ x_2(t_{h+1})\ \cdots\ x_n(t_{h+1})]^T$$
$$= \boldsymbol{\delta}_n[i_1\ i_2 \cdots\ i_n] \cdot [x_1(t_h)\ x_2(t_h)\ \cdots\ x_n(t_h)]^T$$

Of course, it can also written by using the original expression:

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{P} \cdot \boldsymbol{x}(t_h) = [\boldsymbol{\delta}_n^{i_1}\ \boldsymbol{\delta}_n^{i_2} \cdots\ \boldsymbol{\delta}_n^{i_n}] \cdot \boldsymbol{x}(t_h) \quad (28)$$

When the stationary condition $\boldsymbol{x}(t_{h+1}) = \lambda \boldsymbol{x}(t_h)$ is assumed, (28) and (28) can be rewritten as:

$$(\lambda \boldsymbol{I} - \boldsymbol{\delta}_n[i_1\ i_2\ \cdots\ i_n]) \cdot \boldsymbol{x}(t_h) = \boldsymbol{0} \quad (29)$$

and

$$(\lambda \boldsymbol{I} - \boldsymbol{P}) \cdot \boldsymbol{x}(t_h) = \boldsymbol{0}. \quad (30)$$

The above expressions are regarding to the eigenvalues problem of matrix $\boldsymbol{P}$.

With respect to nonnegative matrix $\boldsymbol{P}$, the following theorem has been known.

Perron-Frobenius Theorem. When considering an $n \times n$ nonnegative matrix $\boldsymbol{P}$, we can give the following spectral radius ρ [16, 17]:

$$\rho \geq |\lambda_i|, \quad \forall i \in \{1, 2, \cdots, n\}, \quad (31)$$

where λ_i are eigenvalues of $\boldsymbol{P}$.

***Z*-matrix, *P*-matrix and *M*-matrix.** In some mathematic researchers, the following matrix is defined.

- A real square matrix $\boldsymbol{A}$ is called a Z-matrix if the off diagonal elements are nonpositive [11, 18].
- A real square matrix $\boldsymbol{A}$ is called a P-matrix if each principal minor is positive [11, 19, 20]. Moreover, it is called a P_0-matrix if each principal minor is nonnegative [11, 20].

Based on the above premise, Ostrowski's M-matrix is defined as follows. An M-matrix is a real square matrix $\boldsymbol{A}$ with the following properties [10, 11, 18-24]:

(1) $\boldsymbol{A} = \rho \boldsymbol{I} - \boldsymbol{P}$
$\boldsymbol{P}$: a real square matrix with nonnegative elements,
ρ: a positive number that is larger than the absolute value of all the eigenvalues of $\boldsymbol{P}$. That is, it correspods to the above Z-matrix.

(2) In general, with respect to a real square matrix $\boldsymbol{A}$ with nonpositive off diagonal elements,
(i) there exists $\boldsymbol{x} > \boldsymbol{0}$ that satisfies $\boldsymbol{A}\boldsymbol{x} > 0$;
(ii) $\boldsymbol{A}$ is nonsingular and all the elements of $\boldsymbol{A}^{-1}$ are non-negative;
(iii) the principal minors of $\boldsymbol{A}$ are positive.

From the above theorem, when considering a permutation matrix $\boldsymbol{P}$, spectral radius $\rho = 1$ can be defined. Therefore, matrix $\boldsymbol{I} - \boldsymbol{P}$ for $|\lambda_i| < 1$ is considered to be an M-matrix [25]. Here, it should be noted that complex variables for the eigenvalue and eigenvector cannot be used because the addition of variables is prohibited in the logical systems. In the following numerical examples, a more generalized form of matrix $\boldsymbol{P}$ will be defined. i.e., [9],

$$\tilde{\boldsymbol{P}} = \mathrm{diag}\{\mu_1, \mu_2, \cdots, \mu_n\} \cdot \boldsymbol{P}, \quad (32)$$

where $0 \leq \mu_i < 1$ $(i = 1, 2, \cdots, n)$ are assumed to be failures (or mistakes). In regard to DEDSs with additive disturbances, the stability problem was studied in [26].

In the above considerations, $\boldsymbol{P}$ correspods to a matrix $\boldsymbol{F}_k$ given in (14). Next, we consider recurrent dynamic systems based on (16). That is,

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{F}(\boldsymbol{\delta}_m^k \otimes \boldsymbol{x}(t_h) = \boldsymbol{F} \cdot \begin{bmatrix} \boldsymbol{x}_1(t_h) \\ \boldsymbol{x}_2(t_h) \\ \vdots \\ \boldsymbol{x}_m(t_h) \end{bmatrix}, \quad (33)$$

where $\boldsymbol{x}(\cdot) \in \mathbb{Z}^n$, $\boldsymbol{x}_j(\cdot) \in \mathbb{Z}^n$, $\boldsymbol{F} = [\boldsymbol{P}_1\ \boldsymbol{P}_2\ \cdots\ \boldsymbol{P}_m]$, $\boldsymbol{P}_k \in \boldsymbol{\mathcal{P}} \subset \mathbb{Z}^{n \times n}$ $(k = 1, 2, \cdots, m)$, and

$$\boldsymbol{x}_j(t_h) = \begin{cases} \boldsymbol{x}(t_h) & \text{when } j = k \\ \boldsymbol{0} & \text{when } j \neq k. \end{cases}$$

Since the expression of (33) corresponds to a time-varying system, the relation between h and k should be defined. The above concept will be applied to the following examples.

6. NUMERICAL EXAMPLES

Example 1. This example is a simple recurrent system written below.

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{P}\boldsymbol{x}(t_h), \quad (h = 0, 1, 2, \cdots, 20), \quad (34)$$

where $\boldsymbol{x}(t_h) \in \mathbb{Z}^4$ and [10]

$$\boldsymbol{P} = \boldsymbol{\delta}_4[\ 2\ \ 3\ \ 4\ \ 1\ ] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Figure 5 shows the behavior of responses from $x_1(t_0) = 3$, $x_2(t_0) = 5$, $x_3(t_0) = 7$, $x_4(t_0) = 9$.

[9] Here, it is assumed that there exists $\mu_i \in \mathbb{R}$.
[10] $\boldsymbol{P}$ corresponds to $\boldsymbol{F}_k$, $(m = 1)$ in (14)

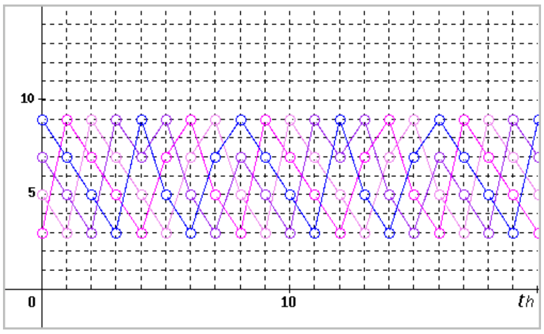

Fig. 5 Sustainable responses for Example 1.

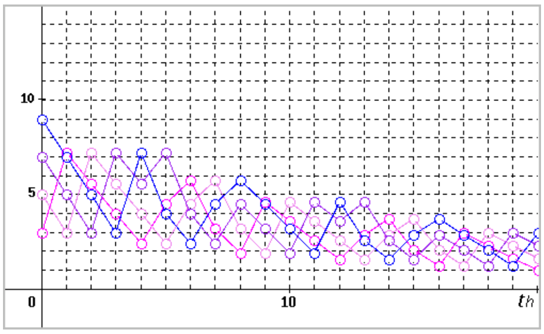

Fig. 6 Decreasing responses for Example 1.

Next, consider the case where some failure (or mistake) occurs in the system as given by (32), i.e.,

$$\tilde{\boldsymbol{P}} = \text{diag}\{\mu,\ 1,\ 1,\ 1\} \cdot \boldsymbol{\delta}_4[\,2\ \ 3\ \ 4\ \ 1\,] \qquad (35)$$

$$= \begin{bmatrix} 0 & 0 & 0 & \mu \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\mu = 0.8$. In this case, the positive real eigenvalue of $\tilde{\boldsymbol{P}}$ becomes $\lambda = 0.946$. Obviously, $|\lambda_i| < 1 \ \ (i = 1, 2, 3, 4)$, that is, $\rho = 1$ can be defined. Thus, it is known that $\boldsymbol{I} - \tilde{\boldsymbol{P}}$ is an M-matrix. The responses become asymptotic as shown in Fig. 6.

Example 2. Consider the following recurrent system based on (33) for $m = 5$:

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{F}\boldsymbol{x}(t_h), \quad (h = 0, 1, 2, \cdots, 20), \qquad (36)$$

where $\boldsymbol{x}(t_h) \in \mathbb{Z}^4$ and $\boldsymbol{F} \in \mathbb{Z}^{4\times 20}$ is given as:

$$\begin{aligned} \boldsymbol{F} = \delta_4[\,2\ \ &4\ \ 1\ \ 3 \\ 3\ \ &4\ \ 2\ \ 1 \\ 4\ \ &1\ \ 2\ \ 3 \\ 3\ \ &1\ \ 4\ \ 2 \\ 4\ \ &3\ \ 1\ \ 2\,] \end{aligned} \qquad (37)$$

by applying (b) $\sim$ (f) in cycles (19), and

$$\boldsymbol{x}_j(t_h) = \begin{cases} \boldsymbol{x}(t_h) & \text{when } j = k \\ \boldsymbol{0} & \text{when } j \neq k. \end{cases}$$

Here, we consider

$$\begin{cases} k = 1 & \text{for } \ h = 0, 1, \\ k = 2 & \text{for } \ h = 2, 3, \\ k = 3 & \text{for } \ h = 4, 5, \\ k = 4 & \text{for } \ h = 6, 7, \\ k = 5 & \text{for } \ h = 8, 9. \end{cases}$$

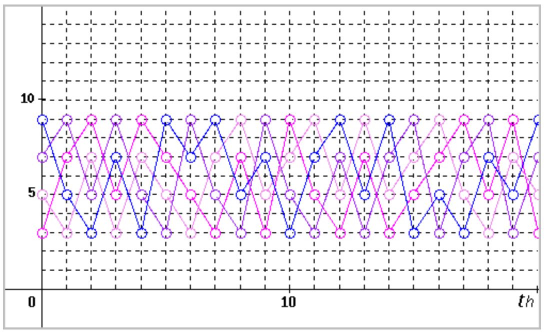

Fig. 7 Sustainable responses for Example 2.

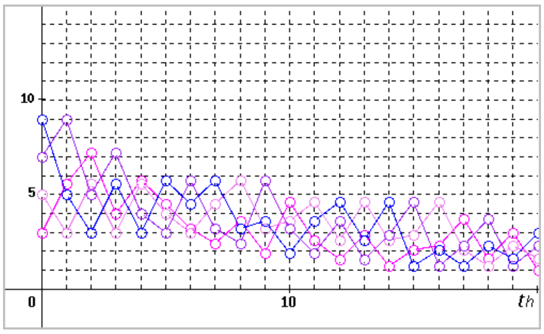

Fig. 8 Decreasing responses for Example 2.

In regard to $h \geq 10$, the above is assumed to be repeated. The responses from $x_1(t_0) = 3$, $x_2(t_0) = 5$, $x_3(t_0) = 7$, and $x_4(t_0) = 9$ are given as shown in Fig. 7. In this case, the responses may be a little chaotic.

Next, consider the case where some failure (or mistake) occurs as given below:

$$\boldsymbol{x}_j(t_h) = \begin{cases} \text{diag}\{\mu, 1, 1, 1\} \cdot \boldsymbol{x}(t_h) & \text{when } j = k \\ \boldsymbol{0} & \text{when } j \neq k. \end{cases}$$

Figure 8 shows the case where $\mu = 0.8$. On the other hand, when the relation beween h and k (as an example) was specified below:

$$\begin{cases} k = 1 & \text{for } \ h = 0, 1, \ \text{and } \mu = 1.1 (\text{initial step } \mu = 1.0), \\ k = 2 & \text{for } \ h = 2, 3, \ \text{and } \mu = 1.1, \\ k = 3 & \text{for } \ h = 4, 5, \ \text{and } \mu = 1.1, \\ k = 4 & \text{for } \ h = 6, 7, \ \text{and } \mu = 0.8, \\ k = 5 & \text{for } \ h = 8, 9, \ \text{and } \mu = 0.8, \end{cases}$$

the reponses become as shown in Fig. 9. Although they may be chaotic, it is known that matrix $\boldsymbol{I} - \tilde{\boldsymbol{P}}$ for

$$\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{P}}_5^2 \tilde{\boldsymbol{P}}_4^2 \tilde{\boldsymbol{P}}_3^2 \tilde{\boldsymbol{P}}_2^2 \tilde{\boldsymbol{P}}_1^2 \qquad (38)$$

is an M-matrix because the positive real eigenvalue of $\tilde{\boldsymbol{P}}$ in (38) becomes 0.968, where,

$$\begin{cases} \tilde{\boldsymbol{P}}_1 = \text{diag}\{1.1, 1, 1, 1\}\boldsymbol{\delta}_4[2\ \ 4\ \ 1\ \ 3], \\ \tilde{\boldsymbol{P}}_2 = \text{diag}\{1.1, 1, 1, 1\}\boldsymbol{\delta}_4[3\ \ 4\ \ 2\ \ 1], \\ \tilde{\boldsymbol{P}}_3 = \text{diag}\{1.1, 1, 1, 1\}\boldsymbol{\delta}_4[4\ \ 1\ \ 2\ \ 3], \\ \tilde{\boldsymbol{P}}_4 = \text{diag}\{0.8, 1, 1, 1\}\boldsymbol{\delta}_4[3\ \ 1\ \ 4\ \ 2], \\ \tilde{\boldsymbol{P}}_5 = \text{diag}\{0.8, 1, 1, 1\}\boldsymbol{\delta}_4[4\ \ 3\ \ 1\ \ 2]. \end{cases}$$

Thus, $\rho = 1$ can be defined. The system will be stable.

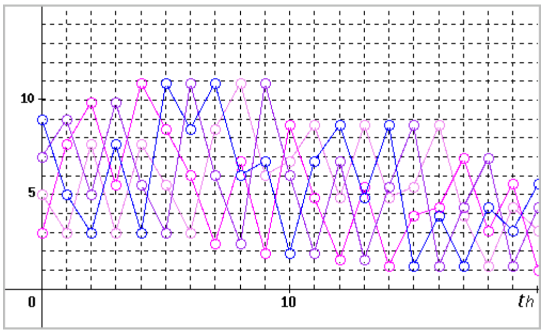

Fig. 9 Decreased-like responses for Example 2.

7. CONCLUSIONS

The security and stability problems of discrete-event dynamic systems (DEDSs) have been studied based on semi-tensor product (STP) and Boolean networks concept. In the analytical process, permutation and logical matrix expressions of such structural nonlinear systems were introduced. The relation to Perron-Frobenius theory and irreducible M-matrix were also described.

It has been said that the theory of descrete event dynamic systems is different from the usual theory of feedback control systems (so-called modern control theory). However, the author thinks that continuous feedback control systems and event-driven dynamic systems should be discussed in a unified theory.

Based on the concept of M-matrix, the author has presented some stability problems of dynamic systems [25, 27, 28]. This paper is considered to be a sequel to those research results.

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