Stability Analysis of Discrete Event Systems Using Multiple Metrics and Simultaneous Linear Inequalities

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Abstract: Finite-state and discrete-event systems are classified into two categories: event-driven and time-driven. In particular, the event-driven types of discrete systems are actually important. There are many those types of systems, e.g., manufacturing systems, industry and welfare robots, computer networks, and so forth. However, the analysis and design of such discrete dynamic systems have not been established, because those systems have severe nonlinear characteristics and do not respond continuously in time. In this paper, the stability of discrete-event dynamic systems are examined using multiple metrics and simultaneous linear inequalities. Numerical examples are shown to clarify the stability and boundedness of periodic discrete-event systems.

Keywords: Finite state systems; event driven; relative stability; discrete control systems; industry robots;

1. INTRODUCTION

There are many finite-state and discrete-event systems in practice that are driven by event signals, e.g., manufacturing systems, industry and welfare robots, computer networks, and so forth. However, the analysis and design of such discrete dynamic systems have not been established, because those systems have severe nonlinear characteristics and do not respond continuously in time. Some authors have attempted to perform stability analysis of finite state and discrete event systems [1-3]. Nevertheless, many of them only discuss and define the stability (e.g., asymptotic, exponential, and Lyapunov stability) for specified discrete event systems.

In this paper, the stability of discrete event systems are examined using multiple metrics and simultaneous linear inequalities. As a result, non-conservative (finite-time) stability conditions in regard to (semi-linear) discrete event systems are derived. In order to clarify the stability and boundedness of discrete event systems, numerical examples are shown in which the nominal systems have a periodic mode.

2. FINITE-STATE AND DISCRETE-EVENT SYSTEMS

In general, finite-state and discrete-event systems can be written as [4, 5]:

\[
\begin{align*}
\{ & x(t_{k+1}) = f(x(t_k), u(t_k)) \\
& y(t_k) = g(x(t_{k+1})) \}, \quad k \in \mathbb{N} := \{0, 1, 2, \ldots \}, \\
& x(t_k) \in \mathbb{Z}^m, \ u(t_k) \in \mathbb{Z}^m, \ y(t_k) \in \mathbb{Z}^n, \\
& f : \mathbb{Z}^n \times \mathbb{Z}^m \rightarrow \mathbb{Z}^n, \ g : \mathbb{Z}^n \rightarrow \mathbb{Z}^n.
\end{align*}
\]

(1)

Although \( \mathbb{Z} \) is considered a finite set of integers, we define the following expression with resolution value \( \gamma \) as follows:

\[ Z_{\gamma} := \{-N\gamma, \ldots, -2\gamma, -\gamma, 0, \gamma, 2\gamma, \ldots, N\gamma \}. \]

Fig. 1 State trajectory of a discrete event system (resolution \( \gamma \) is denoted by \( g \)).

Here, \( Z_{\gamma} \) denotes its positive area, and \( \mathbb{Z}_1 = \mathbb{Z}, \mathbb{Z}_+ = \mathbb{N} \). In the above expression (1), time-driven types of discrete systems are considered in principle, although input sequence \( u(t_k) \) can correspond to an event-sequence vector. Figure 1 shows an example of state (or output) trajectory for an event-sequence \( \{e_1, e_2, e_3, e_4, \ldots \} \). As is obvious, it can be considered that the event sequence corresponds to time sequence \( \{t_1, t_2, t_3, t_4, \ldots \} \). However, the causality relationship between them will be opposite. In addition, they are not one-to-one correspondence. In such systems, the Fourier analysis cannot be applied.

Usually, discrete event systems are represented by state transition graphs. In these examples, the graph becomes as shown in Fig. 2.1. In the mathematical expres-

\[ \text{Fig. 2 State transition graph for a discrete event system.} \]

1In Figs. 1 and 2, the red-arrow transition shows another example of event-sequence.
sion, discrete-event systems are written as follows [2, 6]:
\[ G = (\mathcal{X}, \mathcal{E}, f, \varphi, E_v), \]  
(2)
where \( \mathcal{X} \) is the set of states
\[ x \in \mathcal{X} := \{x_0, x_1, x_2, \ldots, x_N \} \]
and \( \mathcal{E} \) is the set of events
\[ e \in \mathcal{E} := \{e_1, e_2, e_3, \ldots, e_M \}. \]
State transitions are defined by the operators
\[ f_e : \mathcal{X} \rightarrow \mathcal{X}. \]  
(3)
An event \( e \) may only occur if it is in the set defined by the following enable function:
\[ \varphi : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{E}) - \emptyset, \]  
(4)
where \( \mathcal{P}(\mathcal{E}) \) is the power set of \( \mathcal{E} \). The set of all such event trajectories is denoted by \( E_0 \subset \mathcal{E}^N \). In addition, \( E_v \subset E \) represents the set of "valid" event trajectories that are physically realizable.

3. TRANSITION MATRIX REPRESENTATION

In order to study the stability problem, we consider the following semi-linear discrete systems in which the transition matrix is given by integer numbers:
\[ x(t_{k+1}) = \Phi(t_{k+1}, t_k)x(t_k) + f(x(t_k), e(t_k)), \]
\[ \Phi \in \mathbb{Z}^{n \times n}, \ x \in \mathbb{Z}^n, \ f : \mathbb{Z}^n \times \mathbb{Z}^m \rightarrow \mathbb{Z}^n. \]  
(5)
Obviously, when \( \Phi \in \mathbb{Z}^{n \times n} \), the following expression can be written for \( \gamma < 1 \):
\[ \Phi(t_{k+1}, t_k)x(t_k) \in \mathbb{Z}^n, \ x \in \mathbb{Z}^n. \]
In these expressions, the transition matrix \( \Phi(t_k, t_l) \) is considered as follows:
\[ (1) \ \Phi(t_k, t_l)\Phi(t_l, t_l) = \Phi(t_k, t_l) \]
\[ (2) \ \Phi(t_k, t_k) = I. \]
When the nominal system is periodic, it can be written as
\[ (3) \ \Phi(t_{k+p}, t_k) = I, \ p: \text{period.} \]
For each component, it can be expressed as follows:
\[ \Phi(t_k, t_l) = \begin{bmatrix} \phi_{11}(t_k, t_l) & \cdots & \phi_{1n}(t_k, t_l) \\ \vdots & \ddots & \vdots \\ \phi_{n1}(t_k, t_l) & \cdots & \phi_{nn}(t_k, t_l) \end{bmatrix}. \]
By expanding (5), the following equation is given:
\[ x(t_k) = \Phi(t_k, t_0)x(t_0) + \sum_{l=1}^{k} \Phi(t_k, t_l)f(x(t_{l-1}), e(t_{l-1})). \]  
(6)
In this paper, the event driven function \( f \) is simplified as
\[ \varepsilon_{ii}(t_k) = \frac{f_i(x(t_k), e(t_k))}{x_i(t_k)} \in \mathbb{R}. \]  
(7)
In regard to a diagonal matrix, the following expression can be written:
\[ E(t_k) = \begin{bmatrix} \varepsilon_{11}(t_k) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_{nn}(t_k) \end{bmatrix}. \]  
(8)
Thus, (6) can be rewritten as follows:
\[ x(t_k) = \Phi(t_k, t_0)x(t_0) + \sum_{l=1}^{k} \Phi(t_k, t_l)E(t_{l-1})x(t_{l-1}). \]  
(9)
Here, from the definition of \( f_i \), the resolution of \( E(t_k)x(t_k) \in \mathbb{Z}^n_\gamma \)
will not be changed. Note that there are some properties in regard to the resolution of discretized signals.

Properties of Discritized Signals.

(i) Addition: If \( x \in \mathbb{Z}_{\gamma_1} \) and \( y \in \mathbb{Z}_{\gamma_2} (\gamma_1 \leq \gamma_2) \), then \( x + y \in \mathbb{Z}_{\gamma_1} \).
(ii) Multiplication: If \( x \in \mathbb{Z}_{\gamma_1} \) and \( y \in \mathbb{Z}_{\gamma_2} \), then \( x \cdot y \in \mathbb{Z}_{\gamma_1 \cdot \gamma_2} \).
(iii) Division: If \( x \in \mathbb{Z}_{\gamma_1} \) and \( y \in \mathbb{Z}_{\gamma_2} \), there exists \( x/y \notin \mathbb{Z}_{\gamma_1} \) or \( x/y \notin \mathbb{Z}_{\gamma_2} \). That is, \( x/y \in \mathbb{R} \).

With respect to division (iii), the quantization is given by the following program (e.g., C-language) [7]:
\[ q^\dagger = \text{gamma} \ast (\text{double})(\text{int})(q / \text{gamma}); \]
Here, \( \text{gamma} \) is the resolution value \( \gamma \), and \( q, q^\dagger \) are the original signal and the quantized values, respectively.

4. MULTIPLE METRICS AND INEQUALITIES CONDITION

The metric in the state space is usually defined by a scalar value (e.g., Lyapunov function). However, it may lead to a severe condition for the stability of some kind of nonlinear systems. In those cases, the multiple metrics are useful for analyzing the stability of discrete dynamic systems.

Here, considering \( \ell_\infty \) space, we define the following metric (i.e., norm):
\[ \|x_i(t_k)\|_{\ell_\infty} := \sup_{1 \leq l \leq k} |x_i(t_l)| \in \mathbb{Z}_{\gamma_+}, \]  
(10)
or simply the absolute value,
\[ \|x_i(t_k)\|_{\ell_\infty} := |x_i(t_k)| \in \mathbb{Z}_{\gamma_+}. \]  
(11)
When considering multiple metrics, the following vector can be written:
\[
\|x(tk)\|_{\ell_\infty} \geq \|\Theta(tk)\|_{\ell_\infty}
\]
(22) is rewritten as:
\[
(I - \|\Theta(tk)\|_{\ell_\infty}) \|x(tk)\|_{\ell_\infty} \leq \|y(tk)\|_{\ell_\infty}.
\]
(23)
A matrix expression can also be defined as follows:
\[
\|\Psi(tk)\|_{\ell_\infty} = \begin{bmatrix}
\|\psi_{11}(tk)\|_{\ell_\infty} & \cdots & \|\psi_{1n}(tk)\|_{\ell_\infty} \\
\vdots & \ddots & \vdots \\
\|\psi_{n1}(tk)\|_{\ell_\infty} & \cdots & \|\psi_{nn}(tk)\|_{\ell_\infty}
\end{bmatrix} \in \mathbb{Z}_{\geq 0}^{n \times n}.
\]
(13)
In these considerations, the following inequalities are obtained from (9):
\[
\|x(tk)\|_{\ell_\infty} \leq \|\Phi(tk, t_0)x(t_0)\|_{\ell_\infty} + \|\sum_{l=1}^{k} \Phi(tk, t_l)E(t_{l-1})x(t_{l-1})\|_{\ell_\infty} \leq \|\Phi(tk, t_0)\|_{\ell_\infty} \|x(t_0)\|_{\ell_\infty} + \|\sum_{l=1}^{k} \Phi(tk, t_l)E(t_{l-1})\|_{\ell_\infty} \|x(tk)\|_{\ell_\infty},
\]
(14)
Here, the inequality symbol \(\leq\) denotes that each component of the left side vector is less than or equal to each component of the right side one. Moreover, \(|\cdot|\) indicates the absolute value of each component of the vector. Inequality (15) is rewritten as:
\[
\left| I - \sum_{l=1}^{k} \Phi(tk, t_l)E(t_{l-1}) \right|_{\ell_\infty} \|x(tk)\|_{\ell_\infty} \leq \|\Phi(tk, t_0)\|_{\ell_\infty} \|x(t_0)\|_{\ell_\infty}.
\]
(16)
5. RELATIVE STABILITY
In this paper, it is assumed that the trajectories of nominal system \(y(tk) = \Phi(tk, t_0)x(t_0)\) are bounded as
\[
\|y(tk)\|_{\ell_\infty} \leq \|\Phi(tk, t_0)\|_{\ell_\infty} \cdot \|x(t_0)\|_{\ell_\infty} = Y_0,
\]
(17)
where \(Y_0\) is a vector with positive elements.

**Definition 1.**
Based on (17), if \(\|y(tk)\|_{\ell_\infty} \leq Y\) leads to \(\|x(tk)\|_{\ell_\infty} \leq X\) for all \(k \in \mathbb{N}\), the discrete event system is defined as (finite-time) stable in a relative sense [8]. Here, \(Y\) and \(X\) are vectors of some finite (positive) numbers.

**Theorem 1.**
If there exists a vector \(0 \leq X \leq \bar{X}\) by which the following equation holds in regard to a vector \(Y\) with (bounded) positive elements:
\[
(I - \sum_{l=1}^{k} \Phi(tk, t_l)E(t_{l-1}))X = Y \leq \bar{Y},
\]
(18)
the discrete event system is (finite-time) stable in a relative sense.

In other words, the matrix of the left side of (18), i.e.,
\[
A = I - \sum_{l=1}^{k} \Phi(tk, t_l)E(t_{l-1})
\]
(19)
is Ostrowski’s M-matrix [9], the system becomes (finite-time) stable in a relative sense. As is obvious from (19), the non-diagonal elements of \(A\) is non-positive. On the other hand, the diagonal elements of \(A\) will be positive from the condition of M-matrix. The derivation of the (full) conditions is given in the following APPENDIX [6, 10].

**Proof:**
If \(0 \leq A^{-1} \leq C\) can be obtained, (18) is rewritten as
\[
X = A^{-1}Y \leq C \cdot \bar{Y},
\]
(20)
where \(C\) is a matrix with positive elements. In other words, if \(A\) is an M-matrix, inequality (20) is satisfied. Therefore, \(\|y(tk)\|_{\ell_\infty} \leq Y\) leads to \(\|x(tk)\|_{\ell_\infty} \leq X\) for all \(k \in \mathbb{N}\). That is, it can be shown that the discrete event system is stable in a relative sense.

When the nominal system is asymptotically stable, the stability analysis based on (15) and (16) is easily applied to such discrete systems [8]. However, when the nominal system is not asymptotic such as a periodic one, it will be difficult to apply the theorem. In those cases, the stability condition becomes severe, in other words, conservative. Thus, in this paper the following theorem is presented.

**Theorem 2.**
Instead of (19), we consider the following matrix:
\[
A(tk) = I - \|\Theta(tk)\|_{\ell_\infty},
\]
(21)
where \(\Theta(tk) = [\theta_{ij}(tk)] \) and
\[
\theta_{ij}(tk) = \sum_{l=1}^{k} \phi_{ij}(tk, t_l)E(t_{l-1})x(t_{l-1})/|x(tk)|.
\]
If \(A(tk) \ (k \in \mathbb{N})\) is Ostrowski’s M-matrix, the discrete event system is (finite-time) stable in a relative sense.

**Proof:** As for each element, (14) will be given as
\[
\left[\begin{array}{c}
\|x_1(tk)\|_{\ell_\infty} \\
\|x_2(tk)\|_{\ell_\infty} \\
\vdots \\
\|x_n(tk)\|_{\ell_\infty}
\end{array}\right] \leq \left[\begin{array}{c}
\|y_1(tk)\|_{\ell_\infty} \\
\|y_2(tk)\|_{\ell_\infty} \\
\vdots \\
\|y_n(tk)\|_{\ell_\infty}
\end{array}\right]
\]
(22)
\[
\left[\begin{array}{c}
\|\theta_{11}(tk)\|_{\ell_\infty} \\
\|\theta_{12}(tk)\|_{\ell_\infty} \\
\vdots \\
\|\theta_{1n}(tk)\|_{\ell_\infty}
\end{array}\right] \cdots \left[\begin{array}{c}
\|\theta_{n1}(tk)\|_{\ell_\infty} \\
\|\theta_{n2}(tk)\|_{\ell_\infty} \\
\vdots \\
\|\theta_{nn}(tk)\|_{\ell_\infty}
\end{array}\right] \left[\begin{array}{c}
\|x_1(tk)\|_{\ell_\infty} \\
\|x_2(tk)\|_{\ell_\infty} \\
\vdots \\
\|x_n(tk)\|_{\ell_\infty}
\end{array}\right],
\]
where each \(y_i(tk)\) is the output of nominal system that is written by
\[
y_i(tk) = \sum_{j=1}^{n} \phi_{ij}(tk, t_0)x_j(t_0).
\]
Obviously, (22) is rewritten as:
\[
(I - \|\Theta(tk)\|_{\ell_\infty}) \|x(tk)\|_{\ell_\infty} \leq \|y(tk)\|_{\ell_\infty}.
\]
(23)
As was shown in (18) and (19), (23) can be generally written as
\[
\left( I - \| \Theta(t_k) \|_{\ell_\infty} \right) X = A(t_k)X = Y \leq \bar{Y}. \tag{24}
\]
Therefore, if \( A(t_k) \) \((k \in \mathbb{N})\) is an M-matrix, the following expression can be given as shown in (20):
\[
X = A(t_k)^{-1}Y \leq C \cdot \bar{Y}, \tag{25}
\]
where \( C \) is a matrix with positive elements. That is, (23) is written as follows:
\[
\| x(t_k) \|_{\ell_\infty} \leq \left( I - \| \Theta(t_k) \|_{\ell_\infty} \right)^{-1} \| y(t_k) \|_{\ell_\infty}. \tag{26}
\]
Thus, it can be shown that the discrete event system is (finite-time) stable in a relative sense.

6. NUMERICAL EXAMPLES

Example 1. Consider the following third-order system:
\[
\begin{bmatrix}
x_1(t_{k+1}) \\
x_2(t_{k+1}) \\
x_3(t_{k+1})
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 1 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1(t_k) \\
x_2(t_k) \\
x_3(t_k)
\end{bmatrix} +
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
\begin{bmatrix}
x_1(t_k) \\
x_2(t_k) \\
x_3(t_k)
\end{bmatrix}.
\]

The structure matrix is given below:
\[
\Phi(t_{k+1}, t_k) =
\begin{bmatrix}
0 & 1 & 1 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad \Phi(t_{k+2}, t_k) =
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 1 \\
0 & 0 & -1
\end{bmatrix}, \quad \Phi(t_{k+4}, t_k) =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

In this example, the nominal system contains a periodic mode with \( p = 4 \). First, we assume that the resolution value is \( \gamma = 1.0 \).

In order to derive a non-conservative condition, we apply Theorem 2 to this problem. Here, event-driven signals \( e_1, e_2, \) and \( e_3 \) considered here are as shown in Fig. 3. From (21), \( A(t_k) \) in Theorem 2 can be written as:
\[
A(t_k) = I - \| \Theta(t_k) \|_{\ell_\infty} =
\]

Figure 4 shows the trajectories of state-points \((x_1(t_k), x_2(t_k), x_3(t_k))\) and error signals \((x_1(t_k) - y_1(t_k), x_2(t_k) - y_2(t_k), x_3(t_k) - y_3(t_K))\) in the 3D phase space when \( x_1(0) = 10.0, x_2(0) = 20.0, x_3(0) = 1.0 \).

\[
1 - \| \Theta(t_k) \|_{\ell_\infty} > 0
\]
\[
\Delta_2 = (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) > 0
\]
\[
\Delta_3 = (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) > 0
\]
\[
\Delta_4 = (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) > 0
\]
\[
\Delta_5 = (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) > 0
\]

The stability condition for third-order system is given below:
\[
\Delta_1 = 1 - \| \Theta(t_k) \|_{\ell_\infty} > 0
\]
\[
\Delta_2 = (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) > 0
\]
\[
\Delta_3 = (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) > 0
\]
\[
\Delta_4 = (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) > 0
\]
\[
\Delta_5 = (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) (1 - \| \Theta(t_k) \|_{\ell_\infty}) > 0
\]

Figure 6 shows the principal minors of M-matrix: \( \Delta_1(t_k), \Delta_2(t_k), \Delta_3(t_k), \) and \( \Delta_4(t_k), \Delta_5(t_k) \) for (10), and \( \Delta_1(t_k) \), \( \Delta_2(t_k) \), \( \Delta_3(t_k) \) for (11).
Figure 7 Upper bounds of state-points \((\bar{x}_1, \bar{x}_2, \bar{x}_3)\) trajectories: \(x_1(t_k), x_2(t_k), x_3(t_k), \) and \(\hat{x}_1(t_k), \hat{x}_2(t_k), \hat{x}_3(t_k)\).

Figure 8 State-point trajectories in 3D phase space: \((x_1(t_k), x_2(t_k), x_3(t_k)), (x_1(0), x_2(0), x_3(0))\), discretized error of Example 2.

Figure 9 Time series of state-points: \(x_1(t_k), x_2(t_k), \) and \(x_3(t_k)\) of Example 2.

Figure 10 Principal minors of M-matrix: \(\Delta_1(t_k), \Delta_2(t_k), \Delta_3(t_k)\) for (10), and \(\Delta_1(t_k), \Delta_2(t_k), \Delta_3(t_k)\) for (11).

Figure 11 Upper bounds of state-points \((\bar{x}_1, \bar{x}_2, \bar{x}_3)\) trajectories, \(x_1(t_k), x_2(t_k), x_3(t_k), \) and \(\hat{x}_1(t_k), \hat{x}_2(t_k), \hat{x}_3(t_k)\).

7. CONCLUSION

The dynamics and stability of discrete event systems have been studied using multiple metrics. In the analysis, a set of linear inequalities for nonlinear discrete systems and Ostrowski’s M-matrix were considered. As a result, non-conservative (finite-time) stability conditions in regard to (semi-linear) discrete event systems were derived. The validity of the conditions was verified by numerical examples in which the nominal systems have a periodic mode. The author thinks that the considerations based on multiple metrics and simultaneous linear inequalities will be useful for the stability and security analysis of (nonlinear) discrete-dynamical-systems (DDS) in general.

REFERENCES

[7] Y. Okuyama, Discrete Signals, Feedback and Oscil-


**APPENDIX**

Inequalities (18) and (24) is written as

$$AX = Y \leq \bar{Y}.$$  \hspace{1cm} (28)

As for each element, (28) can be expressed as follows:

$$
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}
= 
\begin{bmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n
\end{bmatrix}$$

where the elements of vectors $x_i$ and $y_i$ ($i = 1, 2, \cdots, n$) are non-negative, and all non-diagonal elements of matrix $(a_{ij})$ ($i, j = 1, 2, \cdots, n$) in the left side of (29) are non-positive (i.e., $x_i \geq 0$, $y_i \geq 0$, and $a_{ij} \leq 0$ for $i \neq j$).

The equality part of (29) can be rewritten as

$$
\begin{bmatrix}
    (1) & a_{12} & \cdots & a_{1n} \\
    0 & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
    x_1^{(1)} \\
    x_2^{(1)} \\
    \vdots \\
    x_n^{(1)}
\end{bmatrix}
= 
\begin{bmatrix}
    y_1^{(1)} \\
    y_2^{(1)} \\
    \vdots \\
    y_n^{(1)}
\end{bmatrix}$$

where the first terms are considered $a_{1j}^{(1)} = a_{ij}$, $x_i^{(1)} = x_j$, $y_i^{(1)} = y_i$, and furthermore,

$$
\begin{align*}
    a_{ij}^{(2)} &= \frac{1}{a_{11}} \begin{bmatrix}
    a_{11}^{(1)} & a_{12}^{(1)} \\
    a_{22}^{(1)} & a_{22}^{(2)} \\
    \vdots & \vdots \\
    a_{nn}^{(1)} & a_{nn}^{(2)}
\end{bmatrix}^{-1} \\
    a_{ij}^{(3)} &= \frac{1}{a_{11}} \begin{bmatrix}
    a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(2)} \\
    a_{22}^{(1)} & a_{22}^{(2)} & a_{23}^{(3)} \\
    \vdots & \vdots & \vdots \\
    a_{nn}^{(1)} & a_{nn}^{(2)} & a_{nn}^{(3)}
\end{bmatrix}^{-1} \\
    \vdots & \\
    a_{ij}^{(n)} &= \frac{1}{a_{11}} \begin{bmatrix}
    a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(n-1)} \\
    a_{22}^{(1)} & a_{22}^{(2)} & \cdots & a_{2n}^{(n-1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{nn}^{(1)} & a_{nn}^{(2)} & \cdots & a_{nn}^{(n-1)}
\end{bmatrix}^{-1} \begin{bmatrix}
    a_{11}^{(n-1)} & a_{12}^{(n-1)} & \cdots & a_{1n}^{(n-1)} \\
    a_{22}^{(n-1)} & a_{22}^{(n-1)} & \cdots & a_{2n}^{(n-1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{nn}^{(n-1)} & a_{nn}^{(n-1)} & \cdots & a_{nn}^{(n-1)}
\end{bmatrix}^{-1} \\
    (i, j &= 2, 3, \cdots, n)
\end{align*}
$$

Then, the right side of (30) can be written as

$$
\begin{align*}
    y_1^{(1)} &= y_1 \\
    y_2^{(2)} &= y_2^{(1)} - \frac{a_{12}^{(1)}}{a_{11}^{(1)}} y_1^{(1)} \\
    y_3^{(3)} &= y_3^{(2)} - \frac{a_{22}^{(2)}}{a_{22}^{(2)}} y_2^{(2)} \\
    \vdots & \\
    y_n^{(n)} &= y_n^{(n-1)} - \frac{a_{nn}^{(n-1)}}{a_{nn}^{(n-1)}} y_{n-1}^{(n-1)}.
\end{align*}
$$

Fig. 12 A graphical interpretation of M-matrix.

provided $a_{11}^{(1)} \geq 0$, $a_{22}^{(2)} > 0$, $\cdots$, $a_{n-1}^{(n-1)} > 0$. Obviously, $y_i^{(1)}$ ($j = 1, 2, \cdots, n$) are obtained in order, because the non-diagonal elements of matrix $[a_{ij}]$ are non-positive.

Therefore, it can be seen that these values are non-negative and bounded if each vector $y_i$ is bounded (i.e., $y_i^{(1)} < \infty$, $i = 1, 2, \cdots, n$). In addition, if $a_{1n}^{(n)} > 0$ is satisfied, then $x_n^{(1)} < \infty$, $x_{n-1}^{(1)} < \infty$, $\cdots$, and $x_1^{(1)} < \infty$ are obtained in reverse order. Here, it should be noted that these conditions can be rewritten as:

$$
\begin{align*}
    a_{11}^{(1)} &= \Delta_1 = a_{11}^{(1)} > 0 \\
    a_{22}^{(2)} &= \Delta_2 = \frac{a_{11}^{(1)} a_{12}^{(1)}}{a_{11}^{(1)}} > 0 \\
    a_{33}^{(3)} &= \Delta_3 = \frac{a_{11}^{(1)} a_{12}^{(1)} a_{13}^{(2)}}{a_{11}^{(1)} a_{22}^{(2)}} > 0 \\
    \vdots & \\
    a_{nn}^{(n)} &= \Delta_n = \frac{a_{11}^{(1)} a_{12}^{(1)} \cdots a_{1n}^{(n-1)}}{a_{1n}^{(n-1)}} > 0.
\end{align*}
$$

Thus, if (31) is satisfied, the following expression can be obtained:

$$
X = A^{-1} Y \leq A^{-1} \bar{Y}.
$$

Figure 12 shows a graphical interpretation of M-matrix in the three-dimensional space. The peak point corresponds to the solution of the equality part of (28).